Markov Chain II

Aug 1, 2022

- office hour today 3 – 5 pm
- note 20 for estimation updated
- review integral operation
- special topic session next week

Markov Chain review

- Markov Property $\mathbb{P}(X_{n+1}=j \mid X_n=i)=\mathbb{P}(X_{n+1}=j \mid X_n=i, X_{n-1}, \cdots X_0)$
- State Space $S=\{H,T\}$, $S=\{1,2,\cdots,6\}$
- Transition Probability $P(i,j)=P_{ij}=\mathbb{P}(X_{n+1}=j \mid X_n=i)$
- Balance Equation $\pi P=\pi$ the solution π is invariant dist.
    - Invariant distribution (aka, stationary distribution, steady state distribution
- Irreducible MC has unique Invariant distribution
- Irreducible and aperiodic MC has $\pi_n \to \pi, n \to \infty$
- Long- term fraction of time of state i

invariant distribution

$$\lim_{N\to\infty} \{\frac{1}{N}\mathbb{I}\{\boldsymbol{X}_n=i\}\}=\pi(i)$$

Given an irreducible MC, if it contains self loop, then it is aperiodic

- The reserve is not true.
  - counterexample, random walk on a triangle

aperiodic MC

$\forall i$

$$d(i) := \gcd\{n \geq 1 \mid P^n(i,i) > 0\}$$

$d=1$ then MC is aperiodic

$d>1$ then MC is periodic w/ period d

$\pi_0 = (0, 1, 0)$ $\pi_1 = (1/2, 0, \frac{1}{2})$ $\pi_2 = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$

P =

```
         0    0.5000    0.5000
    0.5000         0    0.5000
    0.5000    0.5000         0
```

$d = \gcd\{2,3,4,5,\cdots\} = 1$

so random walk on a triangle is aperiodic

```
>> P^2

ans =

    0.5000    0.2500    0.2500
    0.2500    0.5000    0.2500
    0.2500    0.2500    0.5000
```

```
>> P^4

ans =

    0.3750    0.3125    0.3125
    0.3125    0.3750    0.3125
    0.3125    0.3125    0.3750
```

```
>> P^6

ans =

    0.3438    0.3281    0.3281
    0.3281    0.3438    0.3281
    0.3281    0.3281    0.3438
```

```
>> P^3

ans =

    0.2500    0.3750    0.3750
    0.3750    0.2500    0.3750
    0.3750    0.3750    0.2500
```

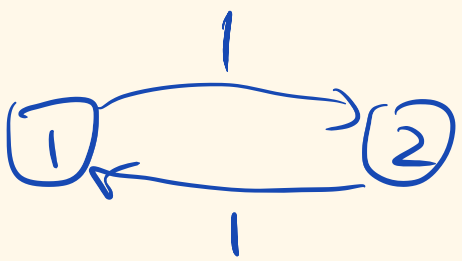
```
>> P^5

ans =

    0.3125    0.3438    0.3438
    0.3438    0.3125    0.3438
    0.3438    0.3438    0.3125
```

```
>> P^7

ans =

    0.3281    0.3359    0.3359
    0.3359    0.3281    0.3359
    0.3359    0.3359    0.3281
```

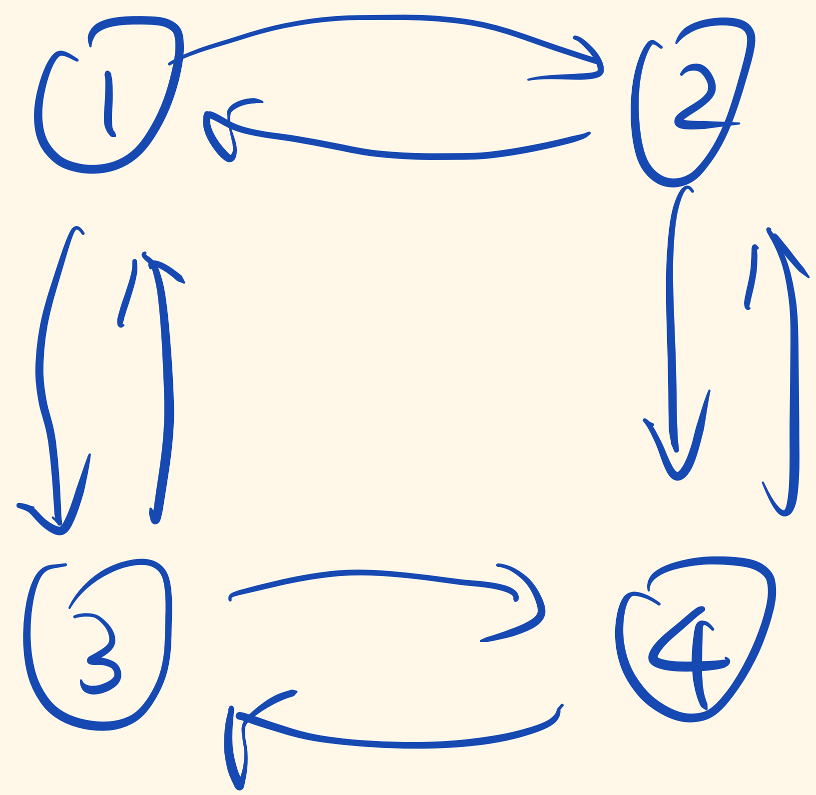
```
>> P^100

ans =

    0.3333    0.3333    0.3333
    0.3333    0.3333    0.3333
    0.3333    0.3333    0.3333
```

Given an irreducible MC, if it is aperiodic, then $\pi_n \to \pi, n \to \infty$

- The reserve is not true
  - Counterexample: random walk on a square



$\pi_0 = [1, 0]$ or $\pi_0 = [\frac{1}{2}, \frac{1}{2}]$
gives different π_n



$d = 2$, $\pi = [\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}]$

start at $\pi_0 = [0.3, 0.25, 0.25, 0.2]$ $\pi_1 = \pi_0 P = \pi$

This MC is periodic but starting at π_0, π is reachable.

```
P =

         0    0.5000         0    0.5000
    0.5000         0    0.5000         0
         0    0.5000         0    0.5000
    0.5000         0    0.5000         0
```

```
>> P^2

ans =

    0.5000         0    0.5000         0
         0    0.5000         0    0.5000
    0.5000         0    0.5000         0
         0    0.5000         0    0.5000
```

```
>> P^3

ans =

         0    0.5000         0    0.5000
    0.5000         0    0.5000         0
         0    0.5000         0    0.5000
    0.5000         0    0.5000         0
```

```
>> P^4

ans =

    0.5000         0    0.5000         0
         0    0.5000         0    0.5000
    0.5000         0    0.5000         0
         0    0.5000         0    0.5000
```

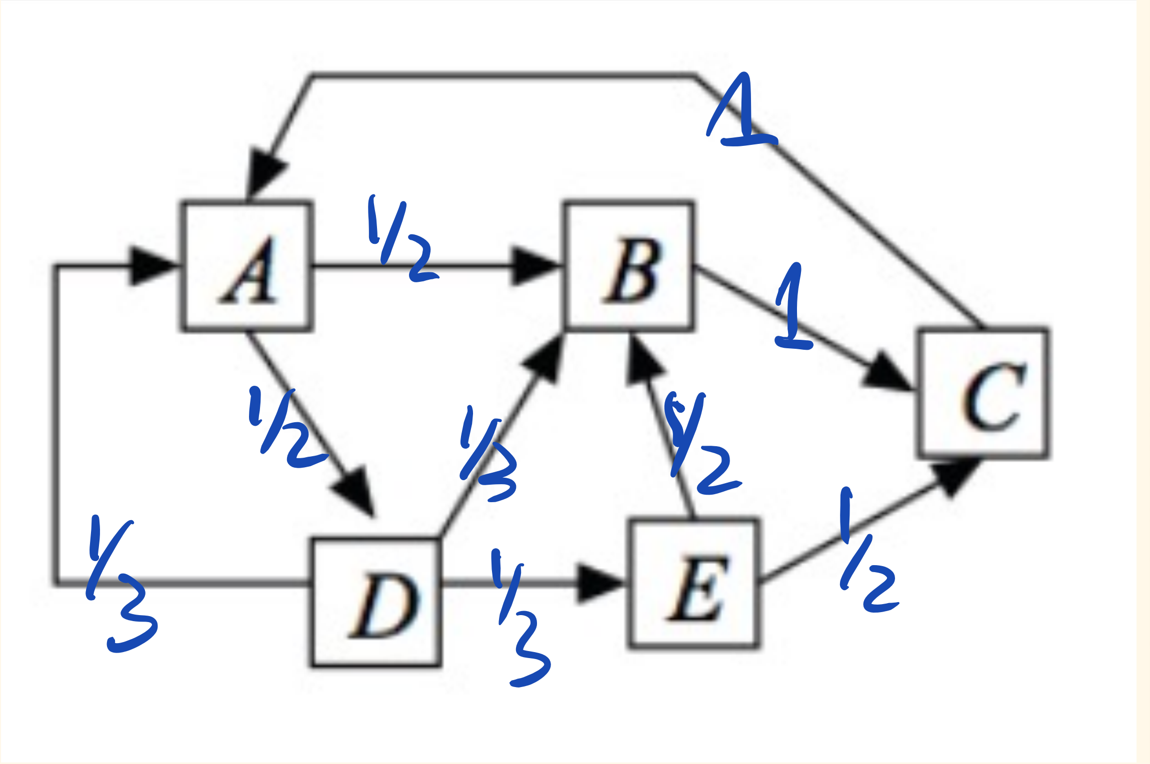
```
>> P^5

ans =

         0    0.5000         0    0.5000
    0.5000         0    0.5000         0
         0    0.5000         0    0.5000
    0.5000         0    0.5000         0
```

$\pi_1 = \pi_0 P$

Example 1

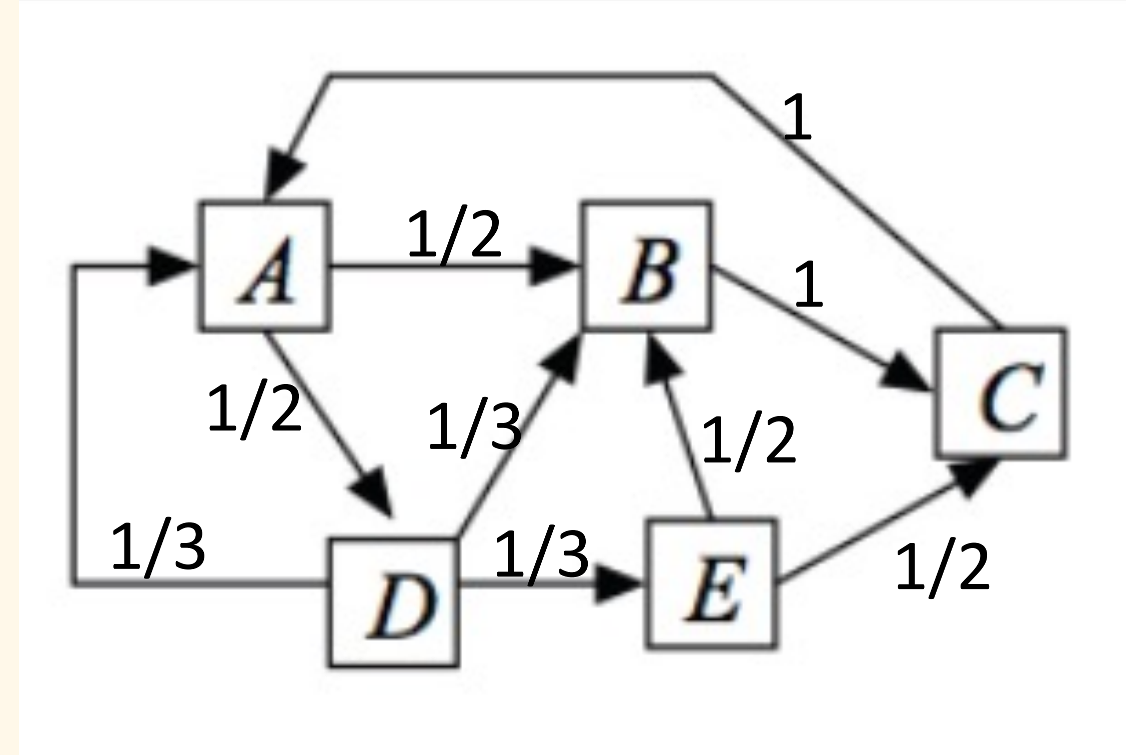
A MC with outgoing arrows are equally likely

1) Is it irreducible? ✓
2) Write transition probability ✓ P_{ij}
3) What's the most frequently visited state?



$$[\pi(1), \pi(2), \pi(3), \pi(4), \pi(5)] \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix} = [\pi(1), \pi(2), \pi(3), \pi(4), \pi(5)]$$

Example 1



A MC with outgoing arrows are equally likely

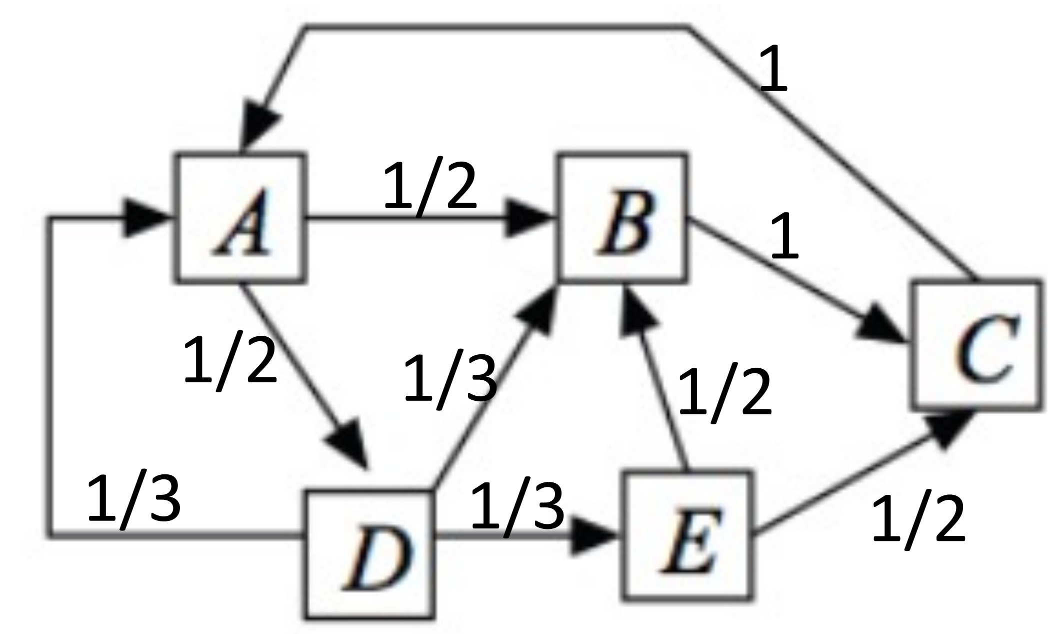
1) Is it irreducible?
2) Write transition probability
3) What's the most frequently visited state?

$$\left.\begin{aligned}
\pi(1) &= \pi(3) + \tfrac{1}{3}\pi(4) \\
\pi(2) &= \tfrac{1}{2}\pi(1) + \tfrac{1}{3}\pi(4) + \tfrac{1}{2}\pi(5) \\
\pi(3) &= \pi(2) + \tfrac{1}{2}\pi(5) \\
\pi(4) &= \tfrac{1}{2}\pi(1) \\
\pi(5) &= \tfrac{1}{3}\pi(4) \\
\pi(1)+\pi(2)+\cdots+\pi(5) &= 1
\end{aligned}\right\} \Rightarrow \pi = \frac{1}{39}(12, 9, 10, 6, 2)$$

$$\max_i \pi(i) = \pi(1)$$

so A is the most frequently visited.

Start at A, how many steps does it take to reach E?



Hitting time of E starting at i is defined as

T_E ← a random variable.

$$\beta(i) := \mathbb{E}(T_E | \boldsymbol{X}_0 = i) \quad \text{for } i = A, B, C, D, E$$

$\beta(C) = 1 + \beta(A)$

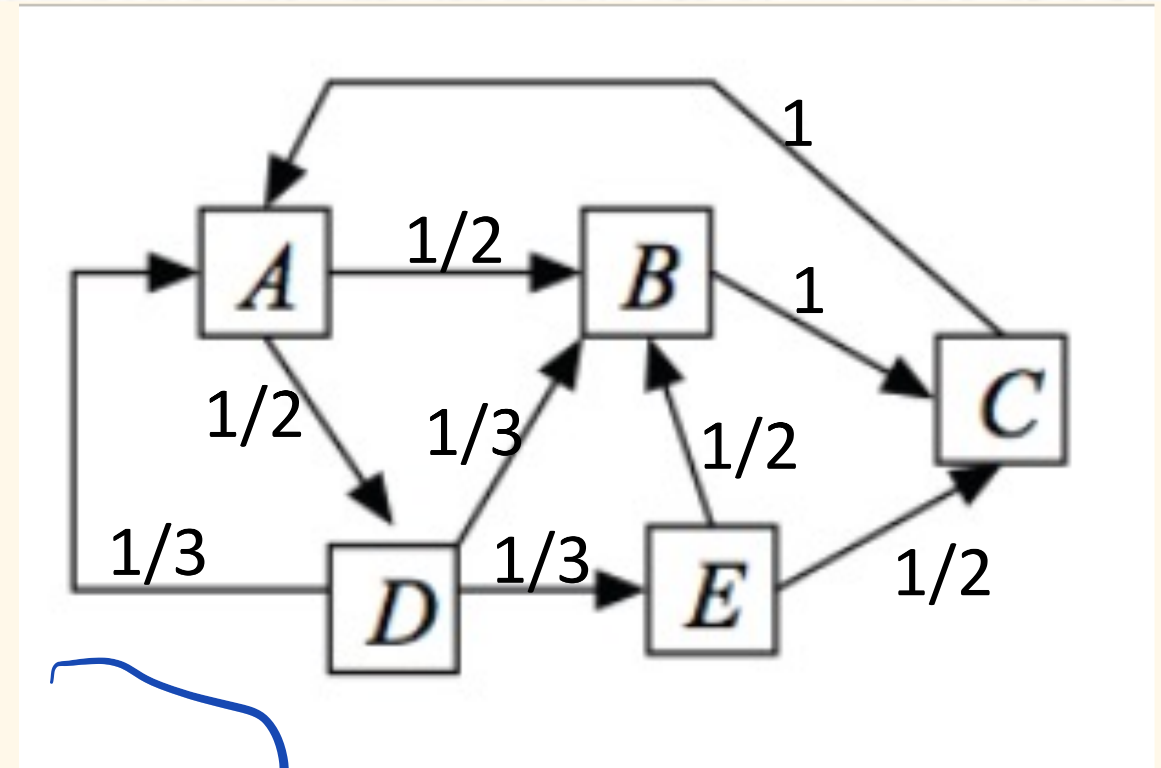
$\beta(B) = 1 + \beta(C)$

$\beta(E) = 0$

Goal: to calculate $\beta(A) := \mathbb{E}(T_E | \boldsymbol{X}_0 = A)$

* $\beta(i)$ is always a function of the goal state. if goal state is not E anymore, FSE needs to be set up again

Hitting time of E starting at i is defined as



$$\beta(i) := \mathbb{E}(T_E | \boldsymbol{X}_0 = i) \quad \text{for } i = A, B, C, D, E$$

$$\beta(A) = 1 + \frac{1}{2}\beta(B) + \frac{1}{2}\beta(D)$$

$$\beta(B) = 1 + \beta(C)$$

$$\beta(C) = 1 + \beta(A)$$

$$\beta(D) = 1 + \frac{1}{3}\beta(A) + \frac{1}{3}\beta(B) + \frac{1}{3}\beta(E)$$

$$\beta(E)\ \mathbb{E}(T_E | X_0 = E) = 0$$

→ first step equation (FSE)

need one step to move from A to B or D.

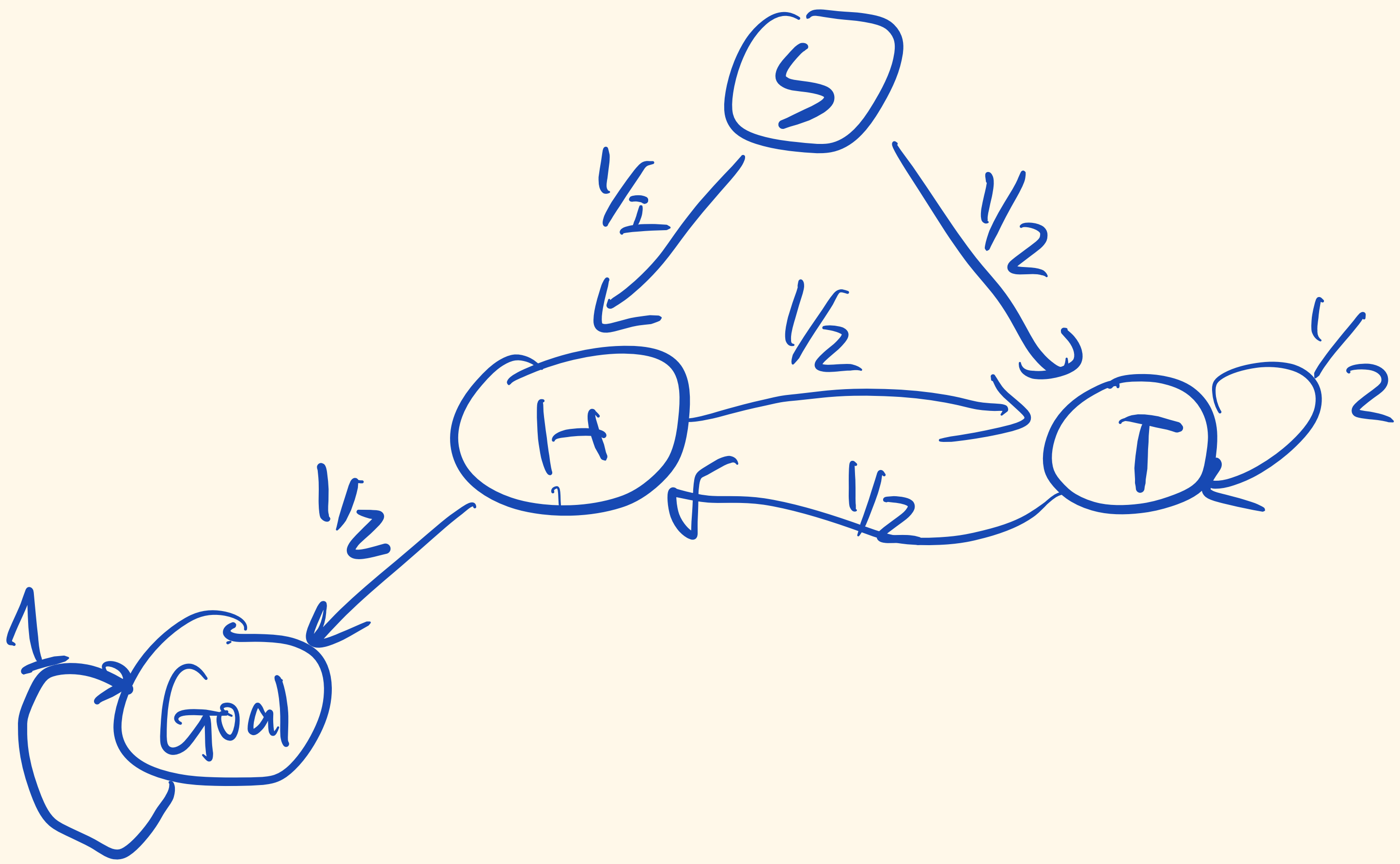
$\beta(A) = 17$, $\beta(B) = 19$, $\beta(C) = 18$, $\beta(D) = 13$, $\beta(E) = 0$

Goal: to calculate $\beta(A) := \mathbb{E}(T_E | \boldsymbol{X}_0 = A)$

Example 2

Flip a fair coin, how many times on average you need to flip to get two head in a row?

solve $\beta(S) = \mathbb{E}(T_{Goal} \mid X_0 = S)$



$$\beta(G) = 0$$

$$\beta(H) = 1 + \frac{1}{2}\beta(G) + \frac{1}{2}\beta(T)$$

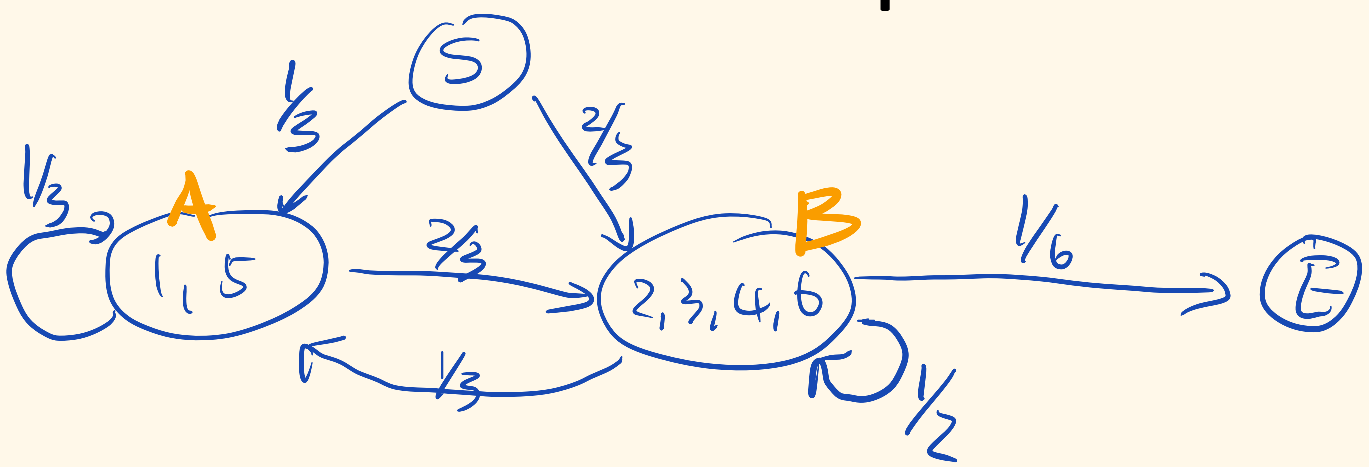
$$\beta(T) = 1 + \frac{1}{2}\beta(H) + \frac{1}{2}\beta(T)$$

$$\boxed{\beta(S)} = 1 + \frac{1}{2}\beta(T) + \frac{1}{2}\beta(H)$$

$$\beta(S) = 6$$

Example 3

Toss a fair 6 face dice, on average, how many times we need to toss until we have the product of two number in a row is 12?



$\beta(S) = \mathbb{E}(T_E \mid X_0 = S)$

$$\beta(E) = 0$$

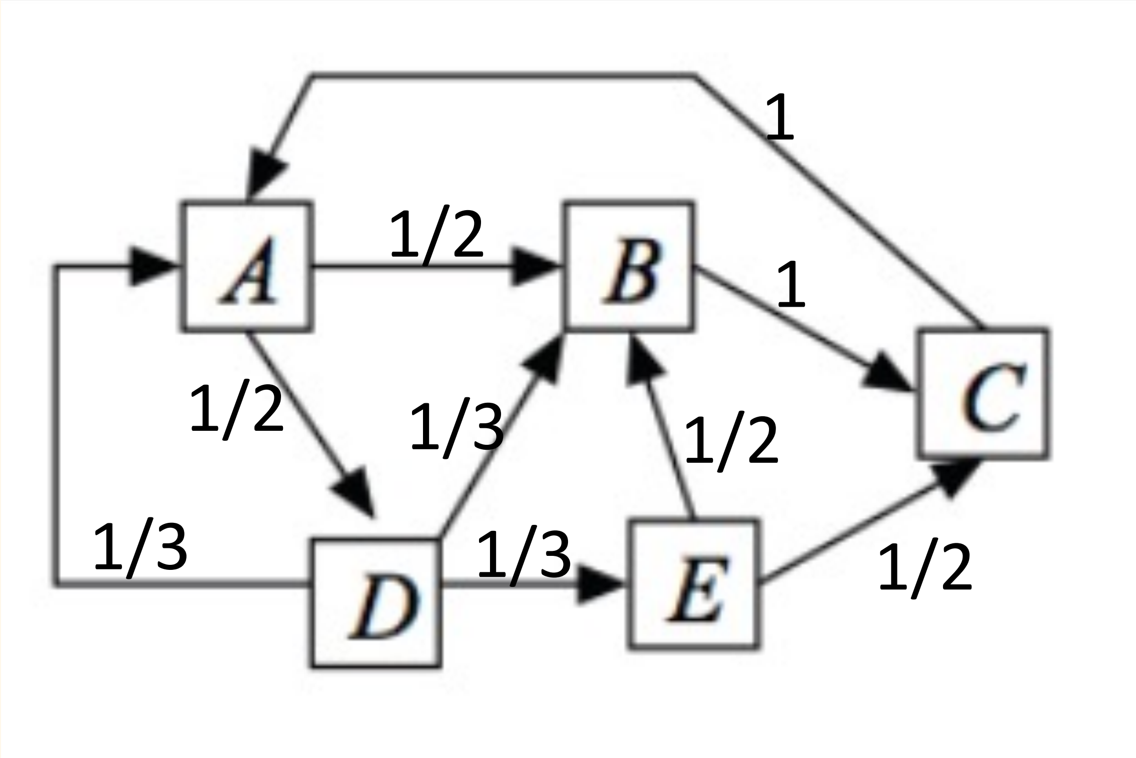
$$\beta(S) = 1 + \tfrac{1}{3}\beta(A) + \tfrac{2}{3}\beta(B)$$

$$\beta(A) = 1 + \tfrac{2}{3}\beta(B) + \tfrac{1}{3}\beta(A)$$

$$\beta(B) = 1 + \tfrac{1}{2}\beta(B) + \tfrac{1}{3}\beta(A) + \tfrac{1}{6}\beta(E)$$

$$\Rightarrow \beta(S) = 10.5 = \frac{21}{2}$$

Example 1.



What's the probability that we start at A and we visit C before we visit E

Define:

arriving at c before E

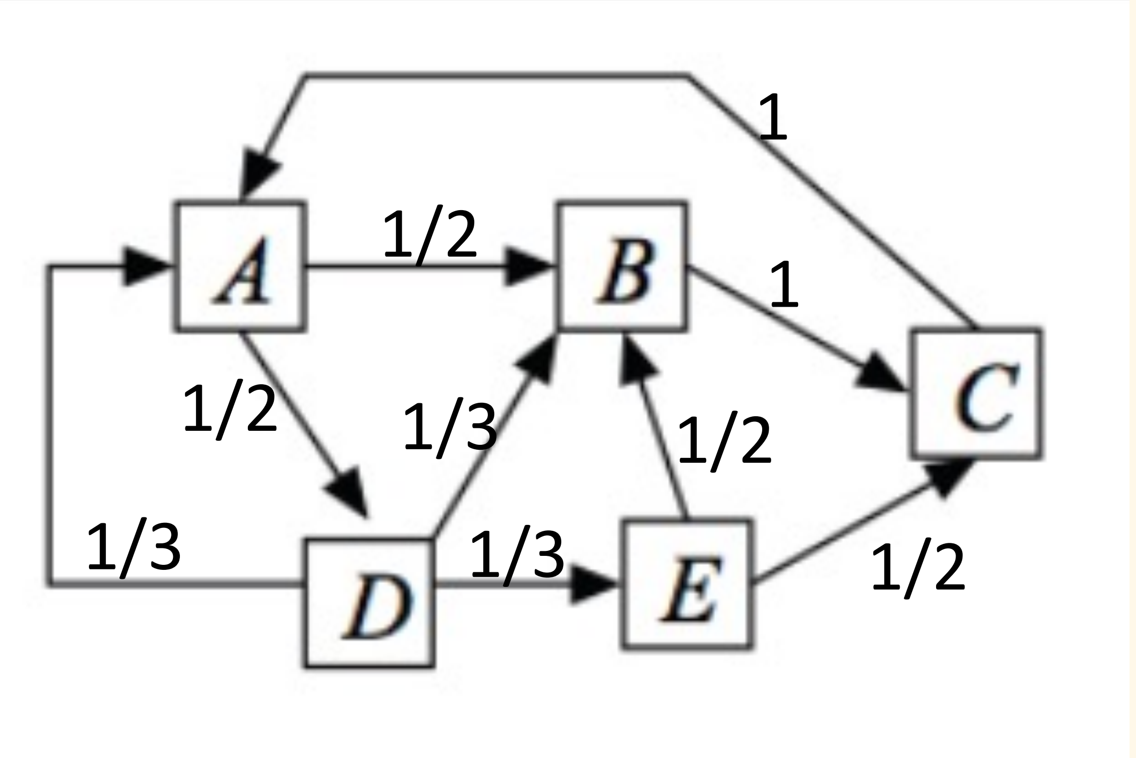
$$\alpha(i) \coloneqq \mathbb{P}(T_c < T_E | \boldsymbol{X}_0 = i) \quad \text{for } i = A, B, C, D, E$$

Goal: to calculate $\alpha(A) \coloneqq \mathbb{P}(T_c < T_E | \boldsymbol{X}_0 = A)$

Example 1

$$\alpha(i) \coloneqq \mathbb{P}(T_c < T_E | \boldsymbol{X}_0 = i)$$

$$\text{for } i = A, B, C, D, E$$



$\alpha(B) = \alpha(C)$

$\alpha(C) = 1$

$\alpha(E) = 0$

$\alpha(A) = \frac{1}{2}\alpha(B) + \frac{1}{2}\alpha(D)$

$\alpha(D) = \frac{1}{3}\alpha(A) + \frac{1}{3}\alpha(B) + \frac{1}{3}\alpha(E)$

→ first step eq.

Goal: to calculate $\alpha(A) \coloneqq \mathbb{P}(T_c < T_E | \boldsymbol{X}_0 = A)$

$\alpha(B) = \alpha(C) = 1$

so $\alpha(A) = \frac{1}{2} + \frac{1}{2}\alpha(D)$

$\alpha(D) = \frac{1}{3}\alpha(A) + \frac{1}{3}$

$\alpha(A) = \frac{1}{2} + \frac{1}{6}\alpha(A) + \frac{1}{6}$ $\times 6$

$5\alpha(A) = 4$

$\alpha(A) = \frac{4}{5}$

General First Step Equation (1)

For a Markov Chain on state space $S = \{1,2, \dots, K\}$ with transition probability P, let T_i be the hitting time of state i.
For a set $A \subset S$ of states, let $T_A = \min\{n \geq 0 | \boldsymbol{X}_n \in A\}$ be the hitting time of the set A.

1) We consider the mean value of T_A

$\beta(i) = \mathbb{E}(T_A | X_0 = i)$, $i \in S$

FSE: $\beta(i) = \begin{cases} 1 + \sum_j P(i,j)\beta(j) & \text{if } i \notin A \\ 0 & \text{if } i \in A \end{cases}$

General First Step Equation (2)

For a Markov Chain on state space $S = \{1,2,\ldots,K\}$ with transition probability P, let T_i be the hitting time of state i.
For a set $A \subset S$ of states, let $T_A = \min\{n \geq 0 | \boldsymbol{X}_n \in A\}$ be the hitting time of the set A.

2) We consider the probability of hitting set A before B

$A, B \subset S$, $A \cap B = \phi$, Let $\alpha(i) = \mathbb{P}(T_A < T_B | X_0 = i)$ $i \in S$

FSE says

$$\alpha(i) = \begin{cases} \sum_j P(i,j)\alpha(j) & \text{if } i \notin A \cup B \\ 1 & \text{if } i \in A \\ 0 & \text{if } i \in B \end{cases}$$

Not required *

General First Step Equation (3)

3) We consider collecting an amount of $h(i)$ every time visiting state i before visiting state A

$$Y = \sum_{n=0}^{T_A} h(\boldsymbol{X}_n)$$

$\gamma(i) := \mathbb{E}(Y \mid X_0 = i) \quad i \in S$

FSE : $\gamma(i) = \begin{cases} h(i) + \sum_j P(i,j)\,\gamma(j), & \text{if } i \notin A \\ h(i) & \text{if } i \in A \end{cases}$

hitting time is the case where $h(i) = 1 \quad \forall i$

Not required *

General First Step Equation (4)

4) We consider a discount factor β for moving one step

$$Z = \sum_{n=0}^{T_A} \beta^n h(\boldsymbol{X}_n)$$

$$g(i) := \mathbb{E}(Z \mid X_0 = i)$$

$$\text{FSE}: \quad g(i) = \begin{cases} h(i) + \beta \sum_j P(i,j)\, g(j) & \text{if } i \notin A \\ h(i) & \text{if } i \in A \end{cases}$$